

# Finite difference/local discontinuous Galerkin method for solving the fractional diffusion-wave equation

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**Abstract:** In this paper a finite difference/local discontinuous Galerkin method for the fractional diffusion-wave equation is presented and analyzed. We first propose a new finite difference method to approximate the time fractional derivatives, and give a semidiscrete scheme in time with the truncation error  $O((\Delta t)^2)$ , where  $\Delta t$  is the time step size. Further we develop a fully discrete scheme for the fractional diffusion-wave equation, and prove that the method is unconditionally stable and convergent with order  $O(h^{k+1} + (\Delta t)^2)$ , where  $k$  is the degree of piecewise polynomial. Extensive numerical examples are carried out to confirm the theoretical convergence rates.

**Key words:** Fractional diffusion-wave equation; Time fractional derivative; Local discontinuous Galerkin method; Stability.

**Mathematics Subject Classification:** 65M12; 65M06; 35S10

## 1 Introduction

Fractional calculus, which might be considered as an extension of classical calculus, attracts much attention in recent decades. Fractional order partial differential equations (FPDEs) have been frequently used to solve many scientific problems in various fields, such as quantitative finance, engineering, biology, chemistry, hydrology, and so on [1, 15, 18, 19, 31, 42, 43].

However, analytical solutions for the majority of fractional partial differential equations, which are too complex and cannot expressed explicitly, are very difficult to be applied in the science and engineering, so it is a good choice to use numerical methods to finding numerical solutions for fractional partial differential equations, and has very important theoretical and practical significance. The existed methods solving the FPDEs include finite difference methods [2, 6, 9, 10, 13, 22, 25, 26, 27, 37, 38, 46, 47, 49], finite element methods[8, 11, 12, 16, 17, 33, 50], spectral methods[3, 23, 24], discontinuous Galerkin methods [51, 52], homotopy perturbation method and the variational method [14, 29, 32, 36, 44, 34, 48].

In this paper we consider the following fractional diffusion-wave equation

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \frac{\partial^2 u(x, t)}{\partial x^2} &= f(x, t), \quad (x, t) \in [a, b] \times [0, T], \\ u(x, 0) &= u_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = u_1(x), \quad x \in [a, b], \end{aligned} \quad (1.1)$$

where  $1 < \alpha < 2$  is a parameter describing the order of the fractional time,  $f, u_0, u_1$  are given smooth functions. We do not pay attention to boundary condition in this paper; hence the

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solution is considered to be either periodic or compactly supported.

The time fractional derivative in the equation (1.1), uses the Caputo fractional partial derivative of order  $\alpha$ , defined as [10]

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{1}{\Gamma(2 - \alpha)} \int_0^t \frac{\partial^2 u(x, s)}{\partial s^2} \frac{ds}{(t - s)^{\alpha-1}}, \quad t > 0, \quad 1 < \alpha < 2, \quad (1.2)$$

where  $\Gamma(\cdot)$  is the Gamma function.

The fractional diffusion-wave equation is obtained by replacing the first- or second-order time derivative of the classical diffusion or wave equation with a fractional derivative of order  $1 < \alpha < 2$ , and can be used to interpolate the diffusion equation and wave equation and model many of the mechanical responses and acoustics accurately.

The rest of this paper is constructed as follows. In the section 2 some basic notations and theoretic results are introduced. Then in section 3 we construct our finite difference/discontinuous Galerkin method for the fractional diffusion-wave equation, and stability and error analysis are given. Numerical results are presented in section 4, and the concluding remarks is included in the final section.

## 2 Notations and auxiliary results

In this section we introduce some notations and definitions that will be used later in the following sections.

Let  $\Omega = [a, b]$  be a finite domain, and a partition is given by

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b,$$

we denote the cell by  $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ , for  $j = 1, \dots, N$ , and the cell lengths  $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ ,  $1 \leq j \leq N$ ,  $h = \max_{1 \leq j \leq N} \Delta x_j$ .

We denote by  $u_{j+\frac{1}{2}}^+$  and  $u_{j+\frac{1}{2}}^-$  the values of  $u$  at  $x_{j+1/2}$ , from the right cell  $I_{j+1}$  and from the left cell  $I_j$ , respectively.

The piecewise-polynomial space  $V_h^k$  is defined as the space of polynomials of the degree up to  $k$  in each cell  $I_j$ , i.e.

$$V_h^k = \{v : v \in P^k(I_j), x \in I_j, j = 1, 2, \dots, N\}.$$

For error estimates, we will be using two projections in one dimension  $[a, b]$ , denoted by  $\mathcal{P}$ , i.e., for each  $j$ ,

$$\int_{I_j} (\mathcal{P}\omega(x) - \omega(x))v(x) = 0, \quad \forall v \in P^k(I_j), \quad (2.1)$$

and special projection  $\mathcal{P}^\pm$ , i.e., for each  $j$ ,

$$\begin{aligned} & \int_{I_j} (\mathcal{P}^+\omega(x) - \omega(x))v(x) = 0, \quad \forall v \in P^{k-1}(I_j), \\ \text{and} \quad & \mathcal{P}^+\omega(x_{j-\frac{1}{2}}^+) = \omega(x_{j-\frac{1}{2}}) \\ & \int_{I_j} (\mathcal{P}^-\omega(x) - \omega(x))v(x) = 0, \quad \forall v \in P^{k-1}(I_j), \\ \text{and} \quad & \mathcal{P}^-\omega(x_{j+\frac{1}{2}}^-) = \omega(x_{j+\frac{1}{2}}). \end{aligned} \quad (2.2)$$

For the above projections  $\mathcal{P}$  and  $\mathcal{P}^\pm$ , we have [5, 35, 40, 41]

$$\|\omega^e\| + h\|\omega^e\|_\infty + h^{\frac{1}{2}}\|\omega^e\|_{\tau_h} \leq Ch^{k+1}, \quad (2.3)$$

where  $\omega^e = \mathcal{P}\omega - \omega$  or  $\omega^e = \mathcal{P}^\pm\omega - \omega$ .

The notations are used: the scalar inner product on  $L^2(D)$  be denoted by  $(\cdot, \cdot)_D$ , and the associated norm by  $\|\cdot\|_D$ . If  $D = \Omega$ , we drop  $D$ . In the present paper we use  $C$  to denote a positive constant which may have a different value in each occurrence.

### 3 The schemes

In this section, we first present a finite difference method to approximate the time fractional derivatives, and then give the implicit fully discrete scheme with space discretized by the local discontinuous Galerkin method. Stability and convergence are detailed analysis.

#### 3.1 Time fractional derivative discretization

We divide the interval  $[0, T]$  uniformly with a time step size  $\Delta t = T/M$ ,  $M \in \mathbb{N}$ ,  $t_n = n\Delta t$ ,  $n = 0, 1, \dots, M$  be the mesh points.

Let  $v(x, t) = \frac{\partial u(x, t)}{\partial t}$ , and from the fact

$$v(x, t_i) = \frac{\partial u(x, t_i)}{\partial t} = \frac{3u(x, t_i) - 4u(x, t_{i-1}) + u(x, t_{i-2})}{2\Delta t} + r_1^n,$$

where the truncation error  $|r_1^n| \leq C(\Delta t)^2$ , we can obtain

$$\begin{aligned} \frac{\partial^\alpha u(x, t_n)}{\partial t^\alpha} &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\partial v(x, s)}{\partial s} \frac{ds}{(t_n - s)^{\alpha-1}} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{\partial v(x, s)}{\partial s} \frac{ds}{(t_n - s)^{\alpha-1}} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{v(x, t_{i+1}) - v(x, t_i)}{\Delta t} \frac{ds}{(t_n - s)^{\alpha-1}} + r_2^n \\ &= \frac{(\Delta t)^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{n-1} b_{n-i-1} \frac{v(x, t_{i+1}) - v(x, t_i)}{\Delta t} + r_2^n \\ &= \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)} [v(x, t_n) + \sum_{i=1}^{n-1} (b_{n-i} - b_{n-i-1})v(x, t_i) - b_{n-1}v(x, t_0)] + r_2^n \\ &= \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)} \left[ \frac{3u(x, t_n) - 4u(x, t_{n-1}) + u(x, t_{n-2})}{2\Delta t} \right. \\ &\quad \left. + \sum_{i=1}^{n-1} (b_{n-i} - b_{n-i-1}) \frac{3u(x, t_i) - 4u(x, t_{i-1}) + u(x, t_{i-2})}{2\Delta t} \right. \\ &\quad \left. - b_{n-1}v(x, t_0) \right] + r_3^n, \end{aligned} \quad (3.1)$$

where

$$b_0 = 1, \quad b_i = (i+1)^{2-\alpha} - i^{2-\alpha}, i = 1, 2, 3, \dots$$

when  $i = 1$ , we take  $u(x, -1) = u(x, 0) - \Delta t u_1(x) + C(\Delta t)^2$  by Taylor expansion.

Similar to the proof in [24], the truncation error  $|r_2^n| \leq C(\Delta t)^{3-\alpha}$ , so  $r_3^n$  satisfied

$$|r_3^n| \leq C(\Delta t)^{3-\alpha}.$$

It is easy to check that

$$\begin{aligned} b_i &> 0, i = 1, 2, \dots, n. \\ 1 &= b_0 > b_1 > b_2 > \dots > b_n, b_n \rightarrow 0 (n \rightarrow \infty). \end{aligned} \tag{3.2}$$

Substituting (3.1) into (1.1), we have

$$\begin{aligned} 3u(x, t_n) - \beta \frac{\partial^2(x, t_n)}{\partial x^2} &= \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) (3u(x, t_i) - 4u(x, t_{i-1}) + u(x, t_{i-2})) \\ &\quad + 2\Delta t b_{n-1} v(x, t_0) + \beta f(x, t_n) + 4u(x, t_{n-1}) \\ &\quad - u(x, t_{n-2}) + \beta r_3^n, \end{aligned}$$

where  $\beta = 2(\Delta t)^\alpha \Gamma(3 - \alpha)$ .

Let  $u^k$  be the numerical approximation to  $u(x, t_k)$ ,  $f^n = f(x, t_n)$ , the problem (1.1) can be discretized by the following scheme

$$\begin{aligned} 3u^n - \beta \frac{\partial^2 u^n}{\partial x^2} &= \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) (3u^i - 4u^{i-1} + u^{i-2}) \\ &\quad + 2\Delta t b_{n-1} v^0 + \beta f^n + 4u^{n-1} - u^{n-2}, \end{aligned} \tag{3.3}$$

where  $u^{-1} = u^0 - \Delta t u_1(x)$ . We know

$$|\beta r_3^n| \leq C(\Delta t)^3,$$

however, by Taylor expansion we have

$$|u(x, t_{-1}) - u(x, 0) + \Delta t u_1(x)| \leq C(\Delta t)^2,$$

therefore the truncation error is  $O(\Delta t)^2$  in scheme (3.3).

### 3.2 Fully discrete schemes

In this subsection we present the fully discrete LDG scheme for the problem (1.1) based on the semidiscrete scheme (3.3).

We rewrite Eq. (1.1) as a first-order system:

$$p = u_x, \quad \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - p_x = f(x, t). \tag{3.4}$$

Let  $u_h^n, p_h^n \in V_h^k$  be the approximations of  $u(\cdot, t_n), p(\cdot, t_n)$ , respectively,  $f^n(x) = f(x, t_n)$ . We define a fully discrete local discontinuous Galerkin scheme as follows: find  $u_h^n, p_h^n \in V_h^k$ , such

that for all test functions  $\phi, w \in V_h^k$ ,

$$\begin{aligned}
& 3 \int_{\Omega} u_h^n \phi dx + \beta \left( \int_{\Omega} p_h^n \phi_x dx - \sum_{j=1}^N ((\widehat{p_h^n \phi^-})_{j+\frac{1}{2}} - (\widehat{p_h^n \phi^+})_{j-\frac{1}{2}}) \right) \\
&= \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_{\Omega} (3u_h^i - 4u_h^{i-1} + u_h^{i-2}) \phi dx + 2\Delta t b_{n-1} \int_{\Omega} v_h^0 \phi dx \\
&\quad + 4 \int_{\Omega} u_h^{n-1} \phi dx - \int_{\Omega} u_h^{n-2} \phi dx + \beta \int_{\Omega} f^n \phi dx, \\
&\int_{\Omega} p_h^n w dx + \int_{\Omega} u_h^n w_x dx - \sum_{j=1}^N ((\widehat{u_h^n w^-})_{j+\frac{1}{2}} - (\widehat{u_h^n w^+})_{j-\frac{1}{2}}) = 0,
\end{aligned} \tag{3.5}$$

The initial conditions  $u_h^{-1}, u_h^0, v_h^0$  are taken as the  $L^2$  projections of  $u(-1), u(0), u_1(0)$ , respectively,

$$\begin{aligned}
\int_{\Omega} u_h^{-1} \phi dx &= \int_{\Omega} \mathcal{P}u(x, -1) \phi dx = \int_{\Omega} u(x, -1) \phi dx, \\
\int_{\Omega} u_h^0 \phi dx &= \int_{\Omega} \mathcal{P}u(x, 0) \phi dx = \int_{\Omega} u_0(x) \phi dx, \\
\int_{\Omega} v_h^0 \phi dx &= \int_{\Omega} \mathcal{P}u_1(x, 0) \phi dx = \int_{\Omega} u_1(x) \phi dx, \quad \forall v \in V_h^k.
\end{aligned} \tag{3.6}$$

The “hat” terms in (3.5) in the cell boundary terms from integration by parts are the so-called “numerical fluxes”, which are single valued functions defined on the edges and should be designed based on different guiding principles for different PDEs to ensure stability. It turns out that we can take the simple choices such that

$$\widehat{u_h^n} = (u_h^n)^-, \quad \widehat{p_h^n} = (p_h^n)^+. \tag{3.7}$$

We remark that the choice for the fluxes (3.7) is not unique. In fact the crucial part is taking  $\widehat{u_h^n}$  and  $\widehat{p_h^n}$  from opposite sides [40, 5].

### 3.3 Stability and Convergence

In order to simplify the notations and without lose of generality, we consider the case  $f = 0$  in its numerical analysis.

**Theorem 3.1.** *For periodic or compactly supported boundary conditions, the fully-discrete LDG scheme (3.5) is unconditionally stable, and there exists a positive constant  $C$  depending on  $u, T, \alpha$ , such that*

$$\|u_h^n\| \leq C(\|u_h^0\| + \Delta t \|u_1(x)\|), \quad n = 1, 2, \dots, M. \tag{3.8}$$

*Proof.* Taking  $\phi = u_h^n, w = \beta p_h^n$  in scheme (3.5), we obtain

$$3\|u_h^n\|^2 + \beta\|p_h^n\|^2 + \beta \sum_{j=1}^N (\Psi(u_h^n, p_h^n)_{j+\frac{1}{2}} - \Psi(u_h^n, p_h^n)_{j-\frac{1}{2}} + \Theta(u_h^n, p_h^n)_{j-\frac{1}{2}})$$

$$\begin{aligned}
&= \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_{\Omega} (3u_h^i - 4u_h^{i-1} + u_h^{i-2}) u_h^n dx + 2\Delta t b_{n-1} \int_{\Omega} v_h^0 u_h^n dx \\
&\quad + 4 \int_{\Omega} u_h^{n-1} u_h^n dx - \int_{\Omega} u_h^{n-2} u_h^n dx,
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
\Psi(u_h^n, p_h^n) &= (p_h^n)^- (u_h^n)^- - \widehat{p_h^n} (u_h^n)^- - \widehat{u_h^n} (p_h^n)^-, \\
\Theta(u_h^n, p_h^n) &= (p_h^n)^- (u_h^n)^- - (p_h^n)^+ (u_h^n)^+ - \widehat{p_h^n} (u_h^n)^- + \widehat{p_h^n} (u_h^n)^+ - \widehat{u_h^n} (p_h^n)^- \\
&\quad + \widehat{u_h^n} (p_h^n)^+.
\end{aligned}$$

If we take the fluxes (3.7), after some manual calculation, we can easily obtain  $\Theta(u_h^n, p_h^n) = 0$ .

Then based on the equation (3.9), we can get

$$\begin{aligned}
3\|u_h^n\|^2 + \beta\|p_h^n\|^2 &= \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_{\Omega} (3u_h^i - 4u_h^{i-1} + u_h^{i-2}) u_h^n dx \\
&\quad + 2\Delta t b_{n-1} \int_{\Omega} v_h^0 u_h^n dx + 4 \int_{\Omega} u_h^{n-1} u_h^n dx \\
&\quad - \int_{\Omega} u_h^{n-2} u_h^n dx \\
&\leq \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) (3\|u_h^i\| + 4\|u_h^{i-1}\| + \|u_h^{i-2}\|) \|u_h^n\| \\
&\quad + 2\Delta t b_{n-1} \|v_h^0\| \|u_h^n\| + 4\|u_h^{n-1}\| \|u_h^n\| \\
&\quad + \|u_h^{n-2}\| \|u_h^n\|,
\end{aligned}$$

that is

$$\begin{aligned}
3\|u_h^n\| &\leq \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) (3\|u_h^i\| + 4\|u_h^{i-1}\| + \|u_h^{i-2}\|) \\
&\quad + 2\Delta t b_{n-1} \|v_h^0\| + 4\|u_h^{n-1}\| \\
&\quad + \|u_h^{n-2}\|.
\end{aligned} \tag{3.10}$$

We will prove the Theorem 3.1 by mathematical induction. When  $n = 1$ , we can obtain

$$3\|u_h^1\| \leq 2\Delta t \|v_h^0\| + 4\|u_h^0\| + \|u_h^{-1}\| \tag{3.11}$$

Notice that

$$\int_{I_j} u_h^{-1} v dx = \int_{I_j} \mathbb{P}(u(x, 0) - \Delta t u_1(x)) v dx = \int_{I_j} u_h^0 v dx - \Delta t \int_{I_j} u_1(x) v dx,$$

for any  $v \in V_h^k$ . Taking  $v = u_h^{-1}$ , we can obtain

$$\begin{aligned}
\|u_h^{-1}\|_{I_j}^2 &= \int_{I_j} u_h^0 u_h^{-1} dx - \Delta t \int_{I_j} u_1(x) u_h^{-1} dx \\
&\leq \|u_h^0\|_{I_j}^2 + \frac{1}{4} \|u_h^{-1}\|_{I_j}^2 + (\Delta t)^2 \|u_1(x)\|_{I_j}^2 + \frac{1}{4} \|u_h^{-1}\|_{I_j}^2,
\end{aligned}$$

summing over  $j$  from 1 to  $N$ , we can get

$$\|u_h^{-1}\| \leq C(\|u_h^0\| + \Delta t \|u_1(x)\|). \quad (3.12)$$

Similar to the proof of (3.12), we can easily obtain

$$\|v_h^0\| \leq \|u_1(x)\|. \quad (3.13)$$

By using (3.11), (3.12) and (3.13), it is easily to know that there exists a positive constant  $C$ , such that

$$\|u_h^1\| \leq C(\|u_h^0\| + \Delta t \|u_1(x)\|). \quad (3.14)$$

Now suppose the following inequality holds

$$\|u_h^m\| \leq C(\|u_h^0\| + \Delta t \|u_1(x)\|), m = 2, 3 \cdots K, \quad (3.15)$$

we need to prove  $\|u_h^{K+1}\| \leq C(\|u_h^0\| + \Delta t \|u_1(x)\|)$ .

Let  $n = K + 1$  in the inequality (3.10), we can obtain

$$\begin{aligned} 3\|u_h^{K+1}\| &\leq \sum_{i=1}^K (b_{K-i} - b_{K+1-i})(3\|u_h^i\| + 4\|u_h^{i-1}\| + \|u_h^{i-2}\|) \\ &\quad + 2\Delta t b_K \|v_h^0\| + 4\|u_h^K\| \\ &\quad + \|u_h^{K-1}\|. \end{aligned}$$

Using (3.12), (3.13) and (3.15), we can obtain the following inequality easily

$$\|u_h^{K+1}\| \leq C(\|u_h^0\| + \Delta t \|u_1(x)\|).$$

This finishes the proof of the stability result.  $\square$

**Theorem 3.2.** *Let  $u(x, t_n)$  be the exact solution of problem (1.1), which is sufficiently smooth such that  $u \in H^{m+1}$  with  $0 \leq m \leq k + 1$ . Let  $u_h^n$  be the numerical solution of the fully discrete LDG scheme (3.5), then there holds the following error estimate:*

$$\|u(x, t_n) - u_h^n\| \leq C(h^{k+1} + (\Delta t)^2), n = 1, \cdots, M, \quad (3.16)$$

where  $C$  is a constant depending on  $u, T, \alpha$ .

*Proof.* By Taylor expansion we know

$$|u(x, t_{-1}) - u(x, 0) + \Delta t u_1(x)| \leq C(\Delta t)^2,$$

here  $C$  is a positive constant depending on  $u$ . Then by using the property (2.3), we can obtain the following estimate which will be used later,

$$\|u(x, t_{-1}) - u_h^{-1}\| \leq C((\Delta t)^2 + h^{k+1}). \quad (3.17)$$

It is easy to verify that the exact solution of PDE (1.1) satisfies

$$\begin{aligned}
& 3 \int_{\Omega} u(x, t_n) \phi dx + \beta \left( \int_{\Omega} p(x, t_n) \phi_x dx - \sum_{j=1}^N ((p(x, t_n) \phi^-)_{j+\frac{1}{2}} - (p(x, t_n) \phi^+)_{j-\frac{1}{2}}) \right) \\
&= \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_{\Omega} (3u(x, t_i) - 4u(x, t_{i-1}) + u(x, t_{i-2})) \phi dx \\
&\quad + 2\Delta t b_{n-1} \int_{\Omega} v(x, t_0) \phi dx + 4 \int_{\Omega} u(x, t_{n-1}) \phi dx \\
&\quad - \int_{\Omega} u(x, t_{n-2}) \phi dx + \beta \int_{\Omega} f(x, t_n) \phi dx + \beta \int_{\Omega} r_3^n \phi dx, \\
&\int_{\Omega} p(x, t_n) w dx + \int_{\Omega} u(x, t_n) w_x dx - \sum_{j=1}^N ((u(x, t_n) w^-)_{j+\frac{1}{2}} - (u(x, t_n) w^+)_{j-\frac{1}{2}}) = 0,
\end{aligned} \tag{3.18}$$

$\forall v, \eta \in H^1(I_j)$ , for  $j = 1, \dots, N$ .

Denote

$$\begin{aligned}
e_u^n &= u(x, t_n) - u_h^n = \mathcal{P}^- e_u^n - (\mathcal{P}^- u(x, t_n) - u(x, t_n)), \\
e_p^n &= p(x, t_n) - p_h^n = \mathcal{P}^+ e_p^n - (\mathcal{P}^+ p(x, t_n) - p(x, t_n)).
\end{aligned} \tag{3.19}$$

Subtracting (3.5) from (3.18), and with the fluxes (3.7) we can obtain the error equation:

$$\begin{aligned}
& 3 \int_{\Omega} e_u^n \phi dx + \beta \left( \int_{\Omega} e_p^n \phi_x dx - \sum_{j=1}^N (((e_p^n)^+ \phi^-)_{j+\frac{1}{2}} - ((e_p^n)^+ \phi^+)_{j-\frac{1}{2}}) \right) \\
&\quad - \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_{\Omega} (3e_u^i - 4e_u^{i-1} + e_u^{i-2}) \phi dx \\
&\quad - 4 \int_{\Omega} e_u^{n-1} \phi dx \\
&\quad + \int_{\Omega} e_u^{n-2} \phi dx + \beta \int_{\Omega} r_3^n \phi dx + \int_{\Omega} e_p^n w dx + \int_{\Omega} e_u^n w_x dx \\
&\quad - \sum_{j=1}^N (((e_u^n)^- w^-)_{j+\frac{1}{2}} - ((e_u^n)^- w^+)_{j-\frac{1}{2}}) = 0.
\end{aligned} \tag{3.20}$$

Using (3.19), the error equation (3.20) can be written as follows:

$$\begin{aligned}
& 3 \int_{\Omega} \mathcal{P}^- e_u^n \phi dx + \beta \left( \int_{\Omega} \mathcal{P}^+ e_p^n \phi_x dx - \sum_{j=1}^N (((\mathcal{P}^+ e_p^n)^+ \phi^-)_{j+\frac{1}{2}} - ((\mathcal{P}^+ e_p^n)^+ \phi^+)_{j-\frac{1}{2}}) \right) \\
&\quad + \int_{\Omega} \mathcal{P}^+ e_p^n w dx + \int_{\Omega} \mathcal{P}^- e_u^n w_x dx - \sum_{j=1}^N (((\mathcal{P}^- e_u^n)^- w^-)_{j+\frac{1}{2}} - ((\mathcal{P}^- e_u^n)^- w^+)_{j-\frac{1}{2}}) \\
&= \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_{\Omega} (3\mathcal{P}^- e_u^i - 4\mathcal{P}^- e_u^{i-1} + \mathcal{P}^- e_u^{i-2}) \phi dx \\
&\quad + 4 \int_{\Omega} \mathcal{P}^- e_u^{n-1} \phi dx - \int_{\Omega} \mathcal{P}^- e_u^{n-2} \phi dx - \beta \int_{\Omega} r_3^n \phi dx
\end{aligned}$$



$$\begin{aligned}
& - \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_{\Omega} (3(\mathcal{P}^- u(x, t_i) - u(x, t_i)) - 4(\mathcal{P}^- u(x, t_{i-1}) - u(x, t_{i-1}))) \\
& + (\mathcal{P}^- u(x, t_{i-2}) - u(x, t_{i-2})) \phi dx - 4 \int_{\Omega} (\mathcal{P}^- u(x, t_{n-1}) - u(x, t_{n-1})) \phi dx \\
& + \int_{\Omega} (\mathcal{P}^- u(x, t_{n-2}) - u(x, t_{n-2})) \phi dx \\
& + 3 \int_{\Omega} (\mathcal{P}^- u(x, t_n) - u(x, t_n)) \phi dx + \beta \left( \int_{\Omega} (\mathcal{P}^+ p(x, t_n) - p(x, t_n)) \phi_x dx \right. \\
& - \sum_{j=1}^N (((\mathcal{P}^+ p(x, t_n) - p(x, t_n)))^+ \phi^-)_{j+\frac{1}{2}} - (((\mathcal{P}^+ p(x, t_n) - p(x, t_n)))^+ \phi^+)_{j-\frac{1}{2}}) \\
& + \int_{\Omega} (\mathcal{P}^+ p(x, t_n) - p(x, t_n)) w dx + \int_{\Omega} (\mathcal{P}^- u(x, t_n) - u(x, t_n)) w_x dx \\
& \left. - \sum_{j=1}^N (((\mathcal{P}^- u(x, t_n) - u(x, t_n)))^- w^-)_{j+\frac{1}{2}} - (((\mathcal{P}^- u(x, t_n) - u(x, t_n)))^- w^+)_{j-\frac{1}{2}} \right). \tag{3.21}
\end{aligned}$$

Taking the test functions  $\phi = \mathcal{P}^- e_u^n, w = \beta \mathcal{P}^+ e_p^n$  in (3.21), using the properties (2.1)-(2.2), then the following equality holds,

$$\begin{aligned}
& 3 \int_{\Omega} (\mathcal{P}^- e_u^n)^2 dx + \beta \int_{\Omega} (\mathcal{P}^+ e_p^n)^2 dx \\
& = \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_{\Omega} (3\mathcal{P}^- e_u^i - 4\mathcal{P}^- e_u^{i-1} + \mathcal{P}^- e_u^{i-2}) \mathcal{P}^- e_u^n dx \\
& + 4 \int_{\Omega} \mathcal{P}^- e_u^{n-1} \mathcal{P}^- e_u^n dx - \int_{\Omega} \mathcal{P}^- e_u^{n-2} \mathcal{P}^- e_u^n dx \\
& - \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_{\Omega} (3(\mathcal{P}^- u(x, t_i) - u(x, t_i)) - 4(\mathcal{P}^- u(x, t_{i-1}) \\
& - u(x, t_{i-1})) + (\mathcal{P}^- u(x, t_{i-2}) - u(x, t_{i-2}))) \mathcal{P}^- e_u^n dx - 4 \\
& \int_{\Omega} (\mathcal{P}^- u(x, t_{n-1}) - u(x, t_{n-1})) \mathcal{P}^- e_u^n dx - \beta \int_{\Omega} r_3^n \mathcal{P}^- e_u^n dx \\
& + \int_{\Omega} (\mathcal{P}^- u(x, t_{n-2}) - u(x, t_{n-2})) \mathcal{P}^- e_u^n dx \\
& + 3 \int_{\Omega} (\mathcal{P}^- u(x, t_n) - u(x, t_n)) \mathcal{P}^- e_u^n dx \\
& + \beta \int_{\Omega} (\mathcal{P}^+ p(x, t_n) - p(x, t_n)) \mathcal{P}^+ e_p^n dx.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\|\mathcal{P}^- e_u^n\| & \leq \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) (3\|\mathcal{P}^- e_u^i\| + 4\|\mathcal{P}^- e_u^{i-1}\| + \|\mathcal{P}^- e_u^{i-2}\|) \\
& + 4\|\mathcal{P}^- e_u^{n-1}\| + \|\mathcal{P}^- e_u^{n-2}\| + \beta \|r_3^n\| \\
& + \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) (3\|\mathcal{P}^- u(x, t_i) - u(x, t_i)\| \\
& + 4\|\mathcal{P}^- u(x, t_{i-1}) - u(x, t_{i-1})\| + \|\mathcal{P}^- u(x, t_{i-2}) - u(x, t_{i-2})\|)
\end{aligned}$$

$$\begin{aligned}
& + \|\mathcal{P}^- u(x, t_{n-2}) - u(x, t_{n-2})\| + 4\|\mathcal{P}^- u(x, t_{n-1}) - u(x, t_{n-1})\| \\
& + 3\|\mathcal{P}^- u(x, t_n) - u(x, t_n)\| + \sqrt{\beta}\|\mathcal{P}^+ p(x, t_n) - p(x, t_n)\|.
\end{aligned} \tag{3.22}$$

We prove the error estimates (3.16) by mathematical induction. First, we consider the case when  $n = 1$ . From (3.20) and (3.23), we know

$$\begin{aligned}
\|\mathcal{P}^- e_u^1\| & \leq 4\|\mathcal{P}^- e_u^0\| + \|e_u^{-1}\| + \beta\|r_3^1\| \\
& + 4\|\mathcal{P}^- u(x, t_0) - u(x, t_0)\| \\
& + 3\|\mathcal{P}^- u(x, t_1) - u(x, t_1)\| + \sqrt{\beta}\|\mathcal{P}^+ p(x, t_1) - p(x, t_1)\|
\end{aligned} \tag{3.23}$$

Notice the facts that

$$\mathcal{P}^- e_u^0 = 0, \quad \|e_u^{-1}\| \leq C(h^{k+1} + (\Delta t)^2), \quad \|r_3^1\| \leq C(\Delta t)^{3-\alpha},$$

and the property (2.3), we can obtain

$$\|\mathcal{P}^- e_u^1\| \leq C(h^{k+1} + (\Delta t)^2). \tag{3.24}$$

Next we suppose the following inequality holds

$$\|\mathcal{P}^- e_u^m\| \leq C(h^{k+1} + (\Delta t)^2), m = 1, 2, \dots, K. \tag{3.25}$$

When  $n = K + 1$ , from the equation (3.23), we can obtain

$$\begin{aligned}
\|\mathcal{P}^- e_u^{K+1}\| & \leq \sum_{i=1}^K (b_{K-i} - b_{K+1-i}) (3\|\mathcal{P}^- e_u^i\| + 4\|\mathcal{P}^- e_u^{i-1}\| + \|\mathcal{P}^- e_u^{i-2}\|) \\
& + 4\|\mathcal{P}^- e_u^K\| + \|\mathcal{P}^- e_u^{K-1}\| + \beta\|r_3^{K+1}\| \\
& + \sum_{i=1}^K (b_{K-i} - b_{K+1-i}) (3\|\mathcal{P}^- u(x, t_i) - u(x, t_i)\| + 4\|\mathcal{P}^- u(x, t_{i-1}) - u(x, t_{i-1})\| \\
& + \|\mathcal{P}^- u(x, t_{i-2}) - u(x, t_{i-2})\|) + 4\|\mathcal{P}^- u(x, t_K) - u(x, t_K)\| \\
& + \|\mathcal{P}^- u(x, t_{K-1}) - u(x, t_{K-1})\| \\
& + 3\|\mathcal{P}^- u(x, t_{K+1}) - u(x, t_{K+1})\| + \sqrt{\beta}\|\mathcal{P}^+ p(x, t_{K+1}) - p(x, t_{K+1})\| \\
& \leq \sum_{i=1}^K (b_{K-i} - b_{K+1-i}) C(h^{k+1} + (\Delta t)^2) \\
& + 4C(h^{k+1} + (\Delta t)^2) + C(h^{k+1} + (\Delta t)^2) + C(\Delta t)^3 \\
& + \sum_{i=1}^K (b_{K-i} - b_{K+1-i}) Ch^{k+1} + Ch^{k+1} + C(\Delta t)^{\frac{\alpha}{2}} h^{k+1}.
\end{aligned}$$

Similar to the proof of (3.24), we can obtain the following result immediately

$$\|\mathcal{P}^- e_u^{K+1}\| \leq C(h^{k+1} + (\Delta t)^2).$$

Thus Theorem 3.2 follows by the triangle inequality and the interpolation property (2.3).  $\square$

## 4 Numerical examples

In this section, we present numerical experiments of the presented finite difference/local discontinuous Galerkin method to the fractional diffusion-wave equation to verify the error estimates in Section 3.

**Example 4.1.** Consider the following fractional diffusion-wave equation

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \frac{\partial^2 u(x, t)}{\partial x^2} &= f(x, t), \quad (x, t) \in [0, 1] \times [0, 1], \\ u(x, 0) &= 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad x \in [0, 1]. \end{aligned} \quad (4.1)$$

Choose a suitable right-hand-side function  $f(x, t)$  such that the exact solution is

$$u(x, t) = t^2 \sin(2\pi x)$$

Table 1-4 display the errors in  $L^2$ -norm and  $L^\infty$ -norm at  $T = 1$  and convergence orders in space for piecewise  $P^k$  polynomials for several values of  $\alpha : 1.2, 1.4, 1.6, 1.8$ , with time step  $\Delta t = 1/1000$ . Obviously the  $(k + 1)$ -th order of accuracy in space are observed, which is in agreement with the theoretic results.

In order to investigate the temporal accuracy of the proposed method, we fix the space step  $h = 1/200$ . Table 5 show that the errors in  $L^2$ -norm and  $L^1$ -norm attain the second-order convergence in time. The results are consistent with our theoretical results in Theorem 3.2.

In Figure 1 we plot the the approximate solution of the three order on the uniform mesh with 100 cells and the exact solution at  $T = 1$  to show the performance of the presented scheme. We can see that the method is very effective and is a good tool to solve such problems.

## 5 Conclusion

In this work, we have presented a finite difference/local discontinuous Galerkin method for the fractional diffusion-wave equation. We first propose a finite difference method to approximate the time fractional derivatives when  $1 < \alpha < 2$ , and then give a fully discrete scheme and prove that the scheme is unconditionally stable and convergent. The extensive numerical example fully confirm the theoretic analysis.

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Table 1: Spatial accuracy test for the time-fractional diffusion-wave equation (4.1) using piecewise  $P^k$  polynomials.  $\alpha = 1.2, \Delta t = \frac{1}{1000}, T = 1$ .

	N	$L^2$ -error	order	$L^\infty$ -error	order
$P^0$	5	0.265109983989909	-	0.623532065154133	-
	10	0.129308265170869	1.03	0.313595441923762	0.98
	20	6.425848539792525E-002	1.01	0.157010259108059	1.00
	40	3.208010396964848E-002	1.00	7.853125916762804E-002	1.00
	80	1.603391954016398E-002	1.00	3.926891940062052E-002	1.00
$P^1$	5	6.736979744152280E-002	-	0.249880240379731	-
	10	1.695495641564284E-002	1.99	6.468476942047330E-002	1.95
	20	4.245128618901225E-003	2.00	1.631233665625631E-002	1.99
	40	1.061671545606701E-003	2.00	4.103801542149732E-003	1.99
	80	2.654420510184321E-004	2.00	1.027598756713433E-003	2.00
$P^2$	5	6.682959934132981E-003	-	3.174066978350254E-002	-
	10	8.506996364720942E-004	2.97	3.971254358826398E-003	3.00
	20	1.068204883018595E-004	2.99	5.116352220441455E-004	2.96
	40	1.336779544959365E-005	3.00	6.443554411463790E-005	2.99
	80	1.672333408099435E-006	3.00	8.069512789853388E-006	3.00

Table 2: Spatial accuracy test for the time-fractional diffusion-wave equation (4.1) using piecewise  $P^k$  polynomials.  $\alpha = 1.4, \Delta t = \frac{1}{1000}, T = 1$ .

	N	$L^2$ -error	order	$L^\infty$ -error	order
$P^0$	5	0.265002133818103	-	0.623263433950011	-
	10	0.129296240434182	1.03	0.313564577675057	0.98
	20	6.425702562008787E-002	1.01	0.157006484631294	1.00
	40	3.207992382929516E-002	1.00	7.853079254830535E-002	1.00
	80	1.603389759999361E-002	1.00	3.926886254925913E-002	1.00
$P^1$	5	6.736429618528465E-002	-	0.249848519443165	-
	10	1.695467033281010E-002	1.99	6.468306250704692E-002	1.95
	20	4.245111564609582E-003	2.00	1.631223051257358E-002	1.99
	40	1.061670492069727E-003	2.00	4.103794461966181E-003	1.99
	80	2.654419852360282E-004	2.00	1.027597816101400E-003	2.00
$P^2$	5	6.682553306760190E-003	-	3.173870305522863E-002	-
	10	8.506873719172307E-004	2.97	3.971185742546351E-003	3.00
	20	1.068201082305646E-004	2.99	5.116330685330400E-004	2.96
	40	1.336778230870020E-005	3.00	6.443547673697195E-005	2.99
	80	1.672322693664752E-006	3.00	8.069510669296185E-006	3.00

Table 3: Spatial accuracy test for the time-fractional diffusion-wave equation (4.1) using piecewise  $P^k$  polynomials.  $\alpha = 1.6, \Delta t = \frac{1}{1000}, T = 1$ .

	N	$L^2$ -error	order	$L^\infty$ -error	order
$P^0$	5	0.264972688983567	-	0.623189941325662	-
	10	0.129291634839670	1.03	0.313552730827037	0.98
	20	6.425644681449535E-002	1.01	0.157004984816329	1.00
	40	3.207985524403432E-002	1.00	7.853061451680375E-002	1.00
	80	1.603389101024813E-002	1.00	3.926884544112283E-002	1.00
$P^1$	5	6.736317907360344E-002	-	0.249838735643862	-
	10	1.695461845352325E-002	1.99	6.468254410369267E-002	1.95
	20	4.245108546494545E-003	2.00	1.631218386012478E-002	1.99
	40	1.061670300965039E-003	2.00	4.103777047141932E-003	1.99
	80	2.654419703069185E-004	2.00	1.027582241539760E-003	2.00
$P^2$	5	6.682478515070153E-003	-	3.173831828892504E-002	-
	10	8.506852141270998E-004	2.97	3.971173620498153E-003	3.00
	20	1.068200376865845E-004	2.99	5.116326863045966E-004	2.96
	40	1.336774817234452E-005	3.00	6.443546414282400E-005	2.99
	80	1.672066664180233E-006	3.00	8.069510170348080E-006	3.00

Table 4: Spatial accuracy test for the time-fractional diffusion-wave equation (4.1) using piecewise  $P^k$  polynomials,  $\alpha = 1.8, \Delta t = \frac{1}{1000}, T = 1$ .

	N	$L^2$ -error	order	$L^\infty$ -error	order
$P^0$	5	0.265395156138938	-	0.624238212597400	-
	10	0.129328509614864	1.03	0.313647188207608	0.98
	20	6.426055758264328E-002	1.01	0.157015597179358	1.00
	40	3.208037034587025E-002	1.00	7.853194655385835E-002	1.00
	80	1.603396389418712E-002	1.00	3.926903380844882E-002	1.00
$P^1$	5	6.736868534645306E-002	-	0.249898961167015	-
	10	1.695486972560303E-002	1.99	6.468562726677418E-002	1.95
	20	4.245123063266223E-003	2.00	1.631242414011957E-002	1.99
	40	1.061671210932901E-003	2.00	4.103841431837951E-003	1.99
	80	2.654420391246725E-004	2.00	1.027635651727810E-003	2.00
$P^2$	5	6.682807553983295E-003	-	3.174004623627680E-002	-
	10	8.506954178648776E-004	2.97	3.971230808741821E-003	3.00
	20	1.068203742828856E-004	2.99	5.116345075529710E-004	2.96
	40	1.336790404833714E-005	3.00	6.443552350912754E-005	2.99
	80	1.673232029466757E-006	3.00	8.070278651961527E-006	3.00

Table 5: Temporal accuracy test for the time-fractional diffusion-wave equation (4.1) using piecewise  $P^2$  polynomials.  $N = 200$

	$\Delta t$	$L^2$ -error	order	$L^1$ -error	order
$\alpha = 1.1$	0.05	2.315895458864733E-006	-	2.078028449959208E-006	-
	0.04	1.452896981500066E-006	2.09	1.301714631618507E-006	2.10
	0.03	8.051939547113251E-007	2.05	7.198086134684791E-007	2.06
	0.02	3.394311162501219E-007	2.13	3.023980936139506E-007	2.14
$\alpha = 1.8$	0.05	1.550654829179151E-004	-	1.396164504933172E-004	-
	0.04	9.400505980906901E-005	2.24	8.464278635512412E-005	2.24
	0.03	5.271466986243190E-005	2.01	4.746819651138549E-005	2.01
	0.02	2.314548304581403E-005	2.03	2.101160952907237E-005	2.01

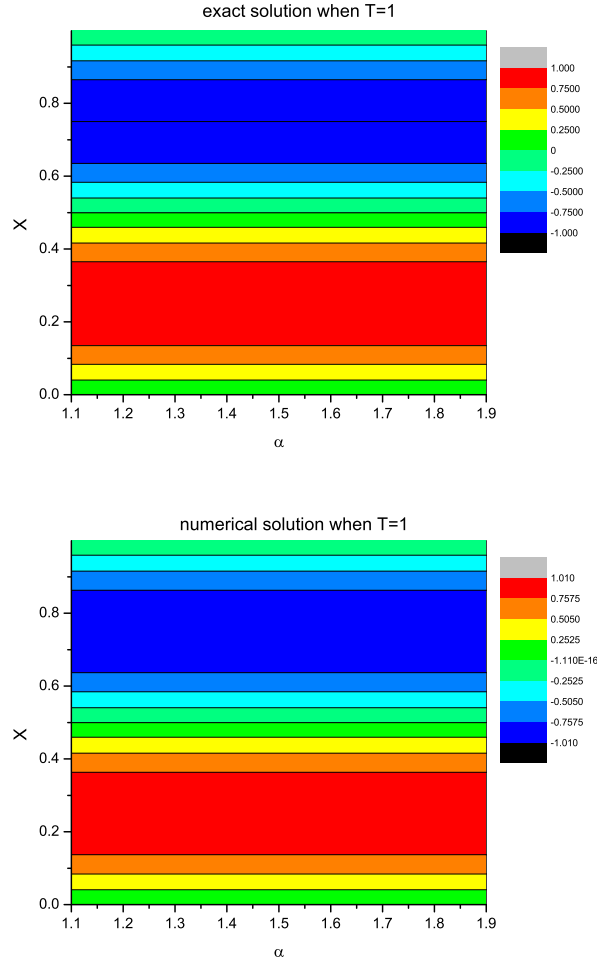


Figure 1: The exact solution is contrasted against approximate solution obtained on the uniform mesh with 100 cells using  $P^2$  elements for Eq.(1.1) when  $T=1$ .

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